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Generator matrix elements for $G_2 \supset SU(3)$: II. Generic representations*

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Abstract. Generator matrix elements are given for G_2 in an $SU(3)$ basis.

1. Introduction

In a recent paper by Farell *et al* (1994), hereafter referred to as I, generator matrix elements of G_2 in an $SU(3)$ basis are given for degenerate representation (one Dynkin label zero) of G_2. In this paper the analysis is extended to generic representations.

In the degenerate case $(a, 0)$ and $(0, b)$ the subgroup provides a complete set of internal labels, so the states are automatically orthogonal and can be normalized straightforwardly; in the generic case (a, b) there is one missing label and it is convenient to use non-orthogonal unnormalized states.

Section 2 deals with the basis states, and discusses their non-orthonormality. In section 3 the generator matrix elements are derived. Section 4 contains some concluding remarks. The earlier paper I contains comments on physical applications of our results and references to previous work on the subject; special reference should be made to the early work of Sviridov *et al* (1975).

2. Basis states

We will use 'character states', for which the integrity basis is provided by the G_2 character generator (see I). It is best to start with the $G_2 \supset SU(3)$ branching rule generating function (Gaskell *et al* 1978)

$$G(A, B; P, Q) = [(1 - AP)(1 - AQ)(1 - BP)(1 - BQ)]^{-1} \times \left[(1 - APQ)^{-1} + B(1 - B)^{-1}\right]. \tag{2.1}$$

We are using Dynkin's labelling of the G_2 fundamental representations: $(1, 0)$ is the 14-plet and $(0, 1)$ the septet. In I and in Gaskell *et al* (1978) that numbering is reversed.

The power series expansion of (2.1) gives the $G_2 \supset SU(3)$ branching rules: the coefficient of $A^a B^b P^p Q^q$ is the multiplicity of the $SU(3)$ representation (p, q) in the G_2 representation (a, b). But equation (2.1) does more than count multiplicities. We interpret

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$AP \sim \lambda$, $AQ \sim \nu^*$, $APQ \sim \alpha$, $BP \sim \eta$, $BQ \sim \zeta^*$, $B \sim \theta$ as the highest states of the $SU(3)$ representations contained in the fundamental G_2 representations (see figure 1 of I); the highest state of any $SU(3)$ representation in any G_2 representation is then given by the appropriate product of powers of them.

We distinguish two types of state, called α-states and θ-states, according to whether the highest state of the $SU(3)$ representation to which they belong contains α or θ (by equation (2.1) it cannot contain both). For α-states we adopt the exponent of ζ^* as the missing label; for θ-states the missing label is the exponent of ν^*. We call the missing label i in both cases. Then a highest α-state is

$$\left| \begin{array}{c} p\ q\ \ i \\ p\ p\ (p+2q)/3 \end{array} \right\rangle_\alpha = \eta^{b-i}\ \zeta^{*i}\ \lambda^{a-q+i}\ \nu^{*a+b-p-i}\ \alpha^{p+q-a-b} \tag{2.2}$$

and a highest θ-state is

$$\left| \begin{array}{c} p\ q\ \ i \\ p\ p\ (p+2q)/3 \end{array} \right\rangle_\theta = \eta^{p-a+i}\ \zeta^{*q-i}\ \theta^{a+b-p-q}\ \lambda^{a-i}\ \nu^{*i}\ . \tag{2.3}$$

We remark that a state for which $p+q=a+b$ may be called, and labelled as, an α-state or a θ-state (α and θ both absent); its i label as an α-state is q less its i label as a θ-state. We suppress the G_2 representation labels (a, b). The internal $SU(3)$ labels are respectively t, m, y with the isospin labels doubled to avoid half-odd values. The ranges of the labels p, q, i are such that the exponents in (2.2) and (2.3) take all non-negative integer values.

States other than the highest ones given by equations (2.2), (2.3) are obtained by applying the $SU(3)$ lowering generators E_{21}, E_{32}, E_{31}. In differential form, suitable for acting on the states (2.2), (2.3) each is given by the sum of the two expressions for it in equations (2.6) and (2.16) of I.

The basis states of the G_2 representation (a, b) are polynomials of degree a in the $(1, 0)$ states and degree b in the $(0, 1)$ states. Thus only stretched representations (representation labels additive) in the direct product of a copies of $(1, 0)$ and b copies of $(0, 1)$ are to be retained. It is known (I) that the elementary unstretched, and therefore unwanted, representations are all of degree 2.

The G_2 character generator is needed in dealing with the unwanted states. Interpreted as describing the integrity basis for general basis states, it tells us that certain pairs of fundamental representation states are incompatible, i.e. never appear multiplied. The incompatible pairs of a particular weight are equal in number to the unwanted states of that weight. Equating each unwanted state to zero allows us to solve for each incompatible pair in terms of pairs that are compatible. When an incompatible pair appears we eliminate it by means of these incompatibility equations.

We have used a version of the character generator in which all fundamental basis states appear in the denominator factors, as opposed to the version of Gaskell and Sharp (1981) in which only exterior states appear in denominators. Since the incompatibility rules characterize the character generator completely we content ourselves with presenting the rules in tabular form. Incompatibilities between $(1, 0)$ and $(0, 1)$ states are shown in table 1. Those between pairs of $(1, 0)$ states are shown in figure 2 of I (ν, ν^*, δ are compatible with all $(1, 0)$ states). The only incompatible pair of $(0, 1)$ states is $\xi\xi^*$. We should mention that δ is the $m = 0$ state of an $SU(2)$ triplet and κ is an $SU(2)$ scalar.

We have determined all the unwanted states and, setting to zero, found the equations by which incompatible pairs are to be eliminated. We give only the replacements that are actually needed in section 3.

Table 1. Incompatibility table. Each incompatible pair of $(1, 0) \times (0, 1)$ states is marked with a cross. The $(0, 1)$ states η, η^* and the $(1, 0)$ states ν, ν^* are compatible with all other states.

	α	α^*	β	β^*	γ	γ^*	κ	δ	λ	λ^*	μ	μ^*
ζ	×				×			×	×		×	
ζ^*		×				×		×		×		×
ξ	×	×		×	×		×	×	×			×
ξ^*	×	×	×			×	×	×		×	×	
θ	×	×	×	×	×	×	×	×				

$(1, 0) \times (0, 1)$ states:

$$
\begin{aligned}
&\alpha\,\zeta = (2/3)^{1/2}\,\kappa\,\eta - (4/3)^{1/2}\,\nu^*\,\xi^* - 6^{-1/2}\,\lambda\,\theta \\
&\gamma\,\xi = 2^{-1/2}\,\delta\,\eta + 6^{-1/2}\,\kappa\,\eta - (4/3)^{1/2}\,\nu^*\,\xi^* \\
&\beta\,\xi^* = -\,\alpha\,\eta^* - (2/3)^{1/2}\,\kappa\,\zeta^* - 6^{-1/2}\,\nu^*\,\theta \\
&\alpha\,\theta = (2/3)^{1/2}\,\lambda\,\zeta^* + (2/3)^{1/2}\,\nu^*\,\eta \\
&\alpha\,\xi^* = -\,\gamma\,\zeta^* - 3^{-1/2}\,\lambda\,\eta \\
&\lambda^*\,\zeta^* = \nu^*\,\eta^* - 2^{-1/2}\,\mu\,\theta \\
&\mu^*\,\zeta^* = 2^{-1/2}\,\lambda\,\theta + \nu^*\,\xi^* \\
&\lambda\,\zeta = \nu\,\eta - 2^{-1/2}\,\mu^*\,\theta \\
&\mu^*\,\xi = -\,\lambda^*\,\eta - \nu^*\,\zeta \\
&\mu\,\xi^* = -\,\lambda\,\eta^* - \nu\,\zeta^* \\
&\kappa\,\theta = \nu^*\,\zeta + \nu\,\zeta^*\,.
\end{aligned}
\tag{2.4}
$$

$(1, 0)^2$ states:

$$
\begin{aligned}
&\beta\,\mu^* = -\,6^{-1/2}\,\kappa\,\nu^* - 2^{-1/2}\,\delta\,\nu^* + 3^{-1/2}\,\mu\,\lambda \\
&\gamma\,\mu = 1/2\,\alpha\,\nu + 2^{-1/2}\,\lambda\,\delta + (12)^{-1/2}\,\mu^*\,\nu^* \\
&\alpha\,\lambda^* = -\,6^{-1/2}\,\nu^*\,\kappa + 2^{-1/2}\,\nu^*\,\delta - 3^{-1/2}\,\lambda\,\mu \\
&\beta\,\gamma = 6^{-1/2}\,\alpha\,\kappa + 2^{-1/2}\,\alpha\,\delta + 1/3\,\lambda\,\nu^* \\
&\lambda\,\kappa = (3/2)^{1/2}\,\alpha\,\nu - 2^{-1/2}\,\mu^*\,\nu^*\,. \\
&\alpha\,\mu = \beta\,\lambda - 3^{-1/2}\,\nu^{*2} \\
&\alpha\,\mu^* = 3^{-1/2}\,\lambda^2 - \gamma\,\nu^* \\
&\mu\,\mu^* = -\,\nu\,\nu^* - \lambda\,\lambda^*\,.
\end{aligned}
\tag{2.5}
$$

$(0, 1)^2$ states:

$$
\xi\,\xi^* = -\,\eta\,\eta^* - \zeta\,\zeta^* - 1/2\,\theta^2\,. \tag{2.6}
$$

It should be noted that although our states correspond one-to-one to all states of all G_2 representations, they still contain admixtures of unwanted states belonging to lower representations. That does not matter for the purpose of computing generator matrix elements, to which we will turn shortly.

Since our states are non-orthonormal, it is preferable to define the matrix element $(i \mid G \mid j)$ of a generator G between two states $\mid i\,\rangle$ and $\mid j\,\rangle$ as the coefficient of $\mid i\,\rangle$

when G acts on $|\, j\,\rangle$, rather than as the overlap $\langle\, i \mid G \mid j\,\rangle$. Matrices so defined can be multiplied in the usual way. The Wigner–Eckart theorem holds for them. Matrices for operators can be diagonalized and their eigenvalues and eigenstates found by the usual standard techniques.

3. The generator matrix elements

We now calculate the generator matrix elements with respect to α-states (highest state given by (2.2)) and θ-states (highest state given by (2.3)). The six significant generators are the components of two $SU(3)$ tensors $G^{(10)}$ and $G^{(01)}$ which transform by the indicated $SU(3)$ representations.

According to the $SU(3)$ Wigner–Eckart theorem the matrix elements of $G^{(10)}$ are given in terms of its reduced matrix elements (double bars) by

$$\left\langle \begin{matrix} p_2\, q_2 & i' \\ t_2\, m_2\, y_2 \end{matrix} \middle| G^{(10)}_{t,m,y} \middle| \begin{matrix} p_1\, q_1 & i \\ t_1\, m_1\, y_1 \end{matrix} \right\rangle$$
$$= \left\langle p_2\, q_2\ \ i' \,\|\, G^{(10)} \,\|\, p_1\, q_1\ \ i \right\rangle \left\langle \begin{matrix} p_1\, q_1 \\ t_1\, m_1\, y_1 \end{matrix} ; \begin{matrix} 1\,0 \\ t\ m\ y \end{matrix} \middle| \begin{matrix} p_2\, q_2 \\ t_2\, m_2\, y_2 \end{matrix} \right\rangle$$
$$\times \left[(p_2+1)(q_2+1)(p_2+q_2+2)/2\right]^{-1/2} . \tag{3.1}$$

The second factor on the right-hand side is an $SU(3)$ Clebsh–Gordan coefficient. A similar formula exists for the matrix elements of $G^{(01)}$, we will see below that the reduced matrix elements of $G^{(01)}$ may be expressed in terms of those of $G^{(10)}$.

We may write

$$G^{(10)}_{0,0,-\frac{2}{3}} \left| \begin{matrix} p\ q\ i \\ p\ p\ (p+2q)/3 \end{matrix} \right\rangle$$
$$= \sum_{i'} \left| \begin{matrix} p+1\ q\ i' \\ p\ p\ (p+2q-2)/3 \end{matrix} \right\rangle A_{i',i} + \sum_{i'} \left| \begin{matrix} p-1\ q+1\ i' \\ p\ p\ (p+2q-2)/3 \end{matrix} \right\rangle B_{i',i}$$
$$+ \sum_{i'} \left| \begin{matrix} p\ q-1\ i' \\ p\ p\ (p+2q-2)/3 \end{matrix} \right\rangle C_{i',i} \tag{3.2}$$

where the $A_{i',i}$, $B_{i',i}$, $C_{i',i}$ are matrix elements of $G^{(10)}_{0,0,-2/3}$ to be determined. We have suppressed a subscript α or θ on the states in (3.2) and, correspondingly, a superscript α or θ on the coefficients $A_{i',i}$, $B_{i',i}$, $C_{i',i}$.

We remark that an ambiguous state ($p+q=a+b$) is transformed by $G^{(10)}$ into an α-state, an ambiguous state or a θ-state. An α-state always goes to an α-state except when $p+q=a+b+1$ when it can also go to an ambiguous state; and a θ-state always goes to a θ-state except when $p+q=a+b-1$ when it can also go to an ambiguous state.

Apply $E_{12}\,E_{23}$ to both side of (3.2). The result is

$$G^{(10)}_{1,1,\frac{1}{3}} \left| \begin{matrix} p\ q\ i \\ p\ p\ (p+2q)/3 \end{matrix} \right\rangle = \sqrt{\frac{(p+1)\,(p+q+2)}{p+2}} \sum_{i'} \left| \begin{matrix} p+1\ q\ i' \\ p\ p\ (p+2q-2)/3 \end{matrix} \right\rangle A_{i',i} . \tag{3.3}$$

The states in the second and third sums on the right-hand side of (3.2) are annihilated and we can read off the allowed values of i' as well as the matrix element $A_{i',i}$. We find, for

α-states, i' can only be i and

$$A^{\alpha}_{i,i} = (-i - p + a + b)\sqrt{\frac{3(2+p)}{(1+p)(2+p+q)}}\,. \tag{3.4}$$

For θ-states i' can be i or $i-1$ and

$$A^{\theta}_{i,i} = (i - p - q + a + b)\sqrt{\frac{2(2+p)}{(1+p)(2+p+q)}} \tag{3.5}$$

$$A^{\theta}_{i-1,i} = i\sqrt{\frac{2(2+p)}{(1+p)(2+p+q)}}\,. \tag{3.6}$$

Next apply E_{23} to both sides of (3.2), after transferring the first sum to the left. The result is

$$\begin{aligned} G^{(10)}_{1,-1,\frac{1}{3}} & \left| \begin{matrix} p\ q\ i \\ p\ p\ (p+2q)/3 \end{matrix} \right\rangle - \sqrt{\frac{p+q+2}{(p+1)(p+2)}} \\ & \times \sum_{i'} A_{i',i}\, E_{21} \left| \begin{matrix} p+1\ q\ i' \\ p+1\ p+1\ (p+2q+1)/3 \end{matrix} \right\rangle \\ & = \sqrt{q+1} \sum_{i'} B_{i',i} \left| \begin{matrix} p-1\ q+1\ i' \\ p-1\ p-1\ (p+2q-2)/3 \end{matrix} \right\rangle . \end{aligned} \tag{3.7}$$

We can read the values of i' and the matrix elements $B_{i',i}$. For α-states i' can be i or $i+1$, and

$$B^{\alpha}_{i,i} = \frac{(i-q+a)(2-i+p+a+b)}{(1+p)\sqrt{1+q}} \tag{3.8}$$

$$B^{\alpha}_{i+1,i} = \frac{(i-b)(1-2i-p+2a+2b)}{(1+p)\sqrt{1+q}}\,. \tag{3.9}$$

For θ-states i' can also be only i or $i+1$, and

$$B^{\theta}_{i,i} = \frac{(1+i+p)(-i-p+a)}{(1+p)\sqrt{1+q}} \tag{3.10}$$

$$B^{\theta}_{i+1,i} = \frac{(-i+a)(2+i+p-q+a+b)}{(1+p)\sqrt{1+q}}\,. \tag{3.11}$$

Finally transfer the first two sums to the left-hand side of (3.2). Only the third sum remains on the right and we can read the allowed values of i' and the matrix elements $C_{i',i}$:

For α-states i' can be i, $i-1$ or $i+1$ and

$$\begin{aligned} C^{\alpha}_{i,i} = \frac{1}{\sqrt{3}(1+p)(1+q)(2+p+q)} & (-4i - 4i^2 + 4i^3 - 6p \\ & -7ip - 2i^2p + 3i^3p - 8p^2 - 4ip^2 + i^2p^2 - 2p^3 - ip^3 \\ & -6q - 3iq - 4i^2q + i^3q - 13pq - 7ipq - 2i^2pq - 8p^2q \\ & -3ip^2q - p^3q - 5q^2 - iq^2 - i^2q^2 - 7pq^2 - 2ipq^2 - 2p^2q^2 \\ & -q^3 - pq^3 + 6b + 2ib - 8i^2b + 6pb - 6i^2pb - ip^2b \end{aligned}$$

$$+2qb+6iqb-2i^2qb+4pqb+3ipqb+p^2qb+2iq^2b+pq^2b$$
$$+2b^2+4ib^2+2pb^2+3ipb^2-2qb^2+iqb^2-pqb^2-q^2b^2$$
$$+6a-3ia-4i^2a+3pa-3ipa-3i^2pa-4p^2a-p^3a+2iqa$$
$$-i^2qa-2pqa+ipqa-2p^2qa-q^2a+iq^2a-pq^2a+7ba+3iba$$
$$+7pba+2ipba-qba+iqba-q^2ba+b^2a+pb^2a+5a^2$$
$$-ia^2+5pa^2-ipa^2+qa^2+pqa^2+2ba^2+2pba^2+a^3+pa^3) \tag{3.12}$$

$$C^{\alpha}_{i+1,i}=\frac{(i-b)(-i-p+a+b)}{\sqrt{3}(1+p)(1+q)(2+p+q)}(1-3i-p-2ip-p^2-iq$$
$$-pq+3b+2pb+qb+3a+2pa+qa) \tag{3.13}$$

$$C^{\alpha}_{i-1,i}=\frac{i(-4+i-p-2q-a-b)(1+i+a)}{\sqrt{3}(1+q)(2+p+q)}\,. \tag{3.14}$$

For θ-states i' takes the values i or $i-1$ and

$$C^{\theta}_{i,i}=\frac{(i-q)(1-i+q+a)(4+i+p+q+a+b)}{\sqrt{2}(1+q)(2+p+q)} \tag{3.15}$$

$$C^{\theta}_{i-1,i}=\frac{i(1+i+p)(-i-p+a)}{\sqrt{2}(1+q)(2+p+q)}\,. \tag{3.16}$$

When one of the states in the matrix element is ambiguous its label (α) or (θ) must be the same as for the other state. If both are ambiguous they should both be given the same label (α) or (θ).

The matrix elements $A^{(\alpha,\theta)}_{i',i}$, $B^{(\alpha,\theta)}_{i',i}$, $C^{(\alpha,\theta)}_{i',i}$ are all we need to obtain the corresponding reduced matrix elements. Also we can get the needed matrix elements of $G^{(01)}$ by following the steps used above for $G^{(10)}$. If we use lowest states (and near lowest) of the $SU(3)$ IR (q,p) instead of highest states (and near highest) of (p,q), and use Hermitian conjugates of all the generators, then apart from some changes in phase, the steps are identical in every detail, with starred and unstarred variables interchanged.

For α-states the results are

$$\langle p+1\;q\;i\,\|G^{(10)}\|\,p\;q\;i\rangle$$
$$=-\langle q\;p+1\;i\,\|G^{(01)}\|\,q\;p\;i\rangle$$
$$=(-i-p+a+b)\sqrt{\frac{3(2+p)(1+q)(3+p+q)}{2}} \tag{3.17}$$

$$\langle p-1\;q+1\;i\,\|G^{(10)}\|\,p\;q\;i\rangle$$
$$=\langle q+1\;p-1\;i\,\|G^{(01)}\|\,q\;p\;i\rangle$$
$$=(i-q+a)(2-i+p+a+b)\sqrt{\frac{(2+q)(2+p+q)}{2(1+p)}} \tag{3.18}$$

$$\begin{aligned}&\left\langle p-1\; q+1\; i+1 \,\middle\|\, G^{(10)} \,\middle\|\, p\; q\; i\right\rangle\\&\quad= \left\langle q+1\; p-1\; i+1 \,\middle\|\, G^{(01)} \,\middle\|\, q\; p\; i\right\rangle\\&\quad= (i-b)\,(1-2i-p+2a+2b)\,\sqrt{\frac{(2+q)\,(2+p+q)}{2\,(1+p)}}\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\left\langle p\; q-1\; i \,\middle\|\, G^{(10)} \,\middle\|\, p\; q\; i\right\rangle\\&\quad= -\left\langle q-1\; p\; i \,\middle\|\, G^{(01)} \,\middle\|\, q\; p\; i\right\rangle\\&\quad= \frac{1}{\sqrt{6\,(1+p)\,(1+q)\,(2+p+q)}}\,(-4i-4i^2+4i^3-6p\\&\quad -7ip-2i^2p+3i^3p-8p^2-4ip^2+i^2p^2-2p^3-ip^3\\&\quad -6q-3iq-4i^2q+i^3q-13pq-7ipq-2i^2pq-8p^2q\\&\quad -3ip^2q-p^3q-5q^2-iq^2-i^2q^2-7pq^2-2ipq^2-2p^2q^2\\&\quad -q^3-pq^3+6b+2ib-8i^2b+6pb-6i^2pb-ip^2b\\&\quad +2qb+6iqb-2i^2qb+4pqb+3ipqb+p^2qb+2iq^2b+pq^2b\\&\quad +2b^2+4ib^2+2pb^2+3ipb^2-2qb^2+iqb^2-pqb^2-q^2b^2\\&\quad +6a-3ia-4i^2a+3pa-3ipa-3i^2pa-4p^2a-p^3a+2iqa\\&\quad -i^2qa-2pqa+ipqa-2p^2qa-q^2a+iq^2a-pq^2a+7ba+3iba\\&\quad +7pba+2ipba-qba+iqba-q^2ba+b^2a+pb^2a+5a^2\\&\quad -ia^2+5pa^2-ipa^2+qa^2+pqa^2+2ba^2+2pba^2+a^3+pa^3)\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left\langle p\; q-1\; i+1 \,\middle\|\, G^{(10)} \,\middle\|\, p\; q\; i\right\rangle\\&\quad= -\left\langle q-1\; p\; i+1 \,\middle\|\, G^{(01)} \,\middle\|\, q\; p\; i\right\rangle\\&\quad= \frac{(i-b)\,(-i-p+a+b)}{\sqrt{6\,(1+p)\,(1+q)\,(2+p+q)}}\,(1-3i-p-2ip-p^2-iq\\&\quad -pq+3b+2pb+qb+3a+2pa+qa)\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\left\langle p\; q-1\; i-1 \,\middle\|\, G^{(10)} \,\middle\|\, p\; q\; i\right\rangle\\&\quad= -\left\langle q-1\; p\; i-1 \,\middle\|\, G^{(01)} \,\middle\|\, q\; p\; i\right\rangle\\&\quad= i\,(-4+i-p-2q-a-b)\,(1+i+a)\,\sqrt{\frac{1+p}{6\,(1+q)\,(2+p+q)}}\,.\end{aligned} \tag{3.22}$$

For θ-states we find for the reduced matrix elements

$$\begin{aligned}&\left\langle p+1\; q\; i \,\middle\|\, G^{(10)} \,\middle\|\, p\; q\; i\right\rangle\\&\quad= -\left\langle q\; p+1\; i \,\middle\|\, G^{(01)} \,\middle\|\, q\; p\; i\right\rangle\\&\quad= (i-p-q+a+b)\,\sqrt{3\,(2+p)\,(1+q)\,(3+p+q)}\end{aligned} \tag{3.23}$$

$$\begin{aligned}\langle p+1\, q\, i-1 \,\| G^{(10)} \|\, p\, q\, i\rangle &= -\langle q\, p+1\, i-1 \,\| G^{(01)} \|\, q\, p\, i\rangle \\ &= i\sqrt{\frac{3\,(2+p)\,(1+q)\,(3+p+q)}{2}}\end{aligned} \tag{3.24}$$

$$\begin{aligned}\langle p-1\, q+1\, i \,\| G^{(10)} \|\, p\, q\, i\rangle &= \langle q+1\, p-1\, i \,\| G^{(01)} \|\, q\, p\, i\rangle \\ &= (1+i+p)\,(-i-p+a)\sqrt{\frac{(2+q)\,(2+p+q)}{2\,(1+p)}}\end{aligned} \tag{3.25}$$

$$\begin{aligned}\langle p-1\, q+1\, i+1 \,\| G^{(10)} \|\, p\, q\, i\rangle &= \langle q+1\, p-1\, i+1 \,\| G^{(01)} \|\, q\, p\, i\rangle \\ &= (-i+a)\,(2+i+p-q+a+b)\sqrt{\frac{(2+q)\,(2+p+q)}{2\,(1+p)}}\end{aligned} \tag{3.26}$$

$$\begin{aligned}\langle p\, q-1\, i \,\| G^{(10)} \|\, p\, q\, i\rangle &= -\langle q-1\, p\, i \,\| G^{(01)} \|\, q\, p\, i\rangle \\ &= (i-q)\,(1-i+q+a)\,(4+i+p+q+a+b)\sqrt{\frac{1+p}{4\,(1+q)\,(2+p+q)}}\end{aligned} \tag{3.27}$$

$$\begin{aligned}\langle p\, q-1\, i-1 \,\| G^{(10)} \|\, p\, q\, i\rangle &= -\langle q-1\, p\, i-1 \,\| G^{(01)} \|\, q\, p\, i\rangle \\ &= i\,(1+i+p)\,(-i-p+a)\sqrt{\frac{1+p}{4\,(1+q)\,(2+p+q)}}\,.\end{aligned} \tag{3.28}$$

4. Concluding remarks

We have used non-orthonormal generic states; that is convenient when there is a missing label. Our results can be compared with those of I by setting one of the G_2 representation a or b labels equal to zero, omitting the 'missing' label i, and replacing the normalization constants $N_{p,q}$ of I by unity.

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